CTP431: Fundamentals of Computer Music

Fourier Series

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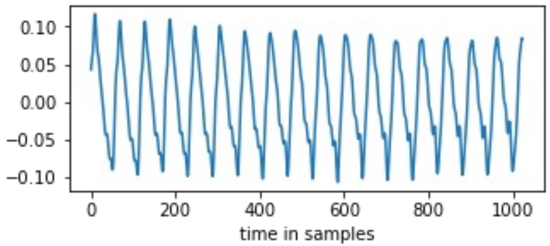
Goals

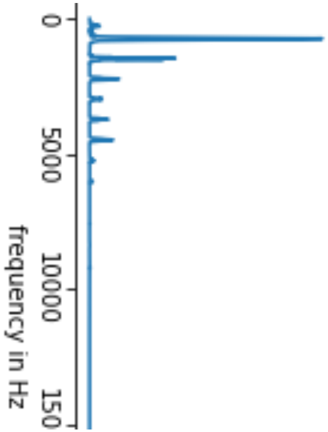
- Representing periodic signals with sinusoids
  - Sinusoids
  - Fourier Series
  - Complex sinusoids
- Additive synthesis

Fourier Series

- A periodic waveform can be represented with a finite set of sinusoids
  - We can find the frequency components of the signal



$x(t)$



This image is from Pink Floyd's "The Dark Side of the Moon"

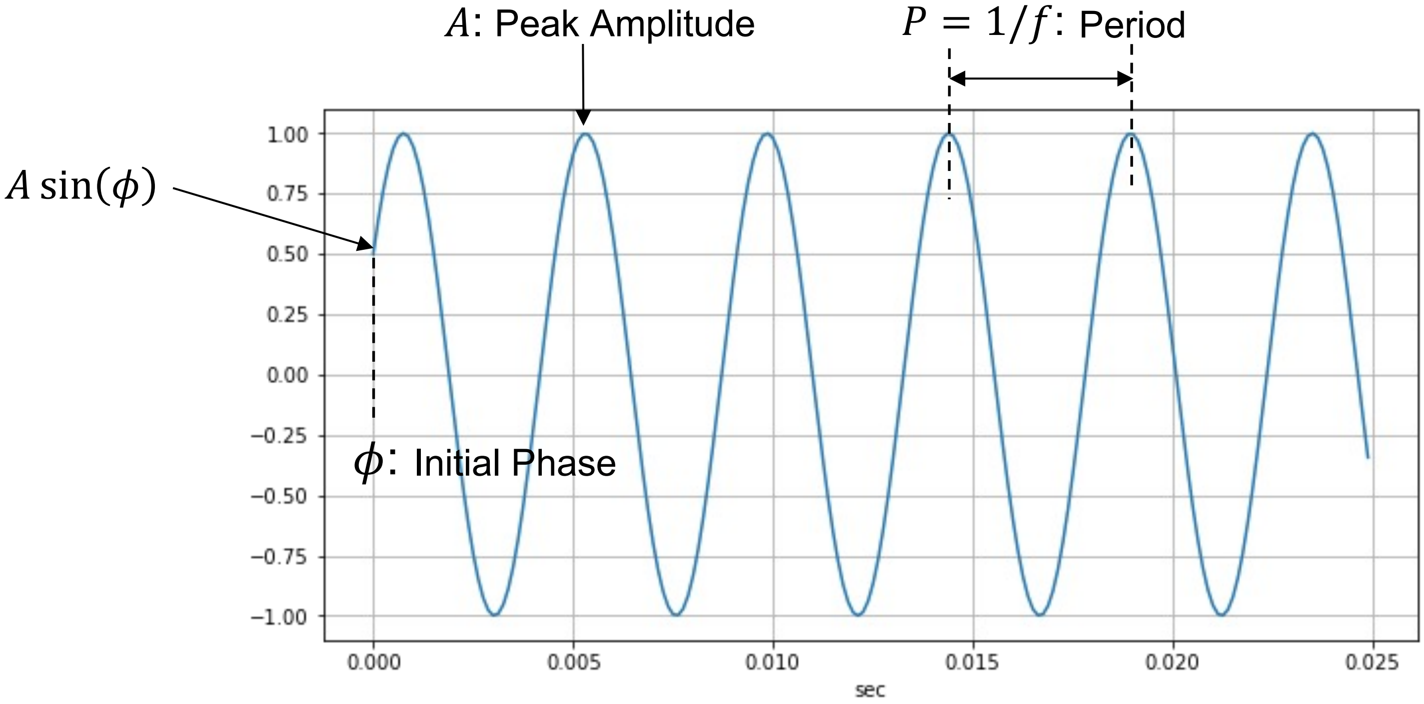
Sinusoids

- A sinusoid is any function having the following form:

$$x(t) = A\sin(\omega t + \phi) = A\sin(2\pi f t + \phi)$$

  - A: peak **amplitude**
  - $\omega(= 2\pi/P = 2\pi f)$: angular **frequency** (rad/sec)
    - P: period (sec)
    - f: frequency (1/sec or Hz)
  - ϕ : initial **phase** (rad)
  - $\omega t + \phi$: instantaneous phase

Plotting Sinusoids



$$x(t) = \sin(2\pi \cdot f \cdot t + \phi)$$

$f = 220\ (Hz)$

$\phi = \pi/6$

In-phase and Quadrature Sinusoidal Components

- A sinusoid with an arbitrary initial phase can be expressed with the sum of a sine function with phase zero (**"in-phase" component**) and a cosine function with phase zero (**"quadrature" component**)

$$x(t) = A\sin(\omega t + \phi) = A\sin(\omega t)\cos(\phi) + A\sin(\phi)\cos(\omega t)$$

$$\cos(\omega t) = \sin(\omega t + \pi/2)$$

$$= [A\cos(\phi)]\underbrace{\sin(\omega t)}_{\text{in-phase component}} + [A\sin(\phi)]\underbrace{\cos(\omega t)}_{\text{quadrature component}}$$

$$= A_1\cos(\omega t) + A_2\sin(\omega t)$$

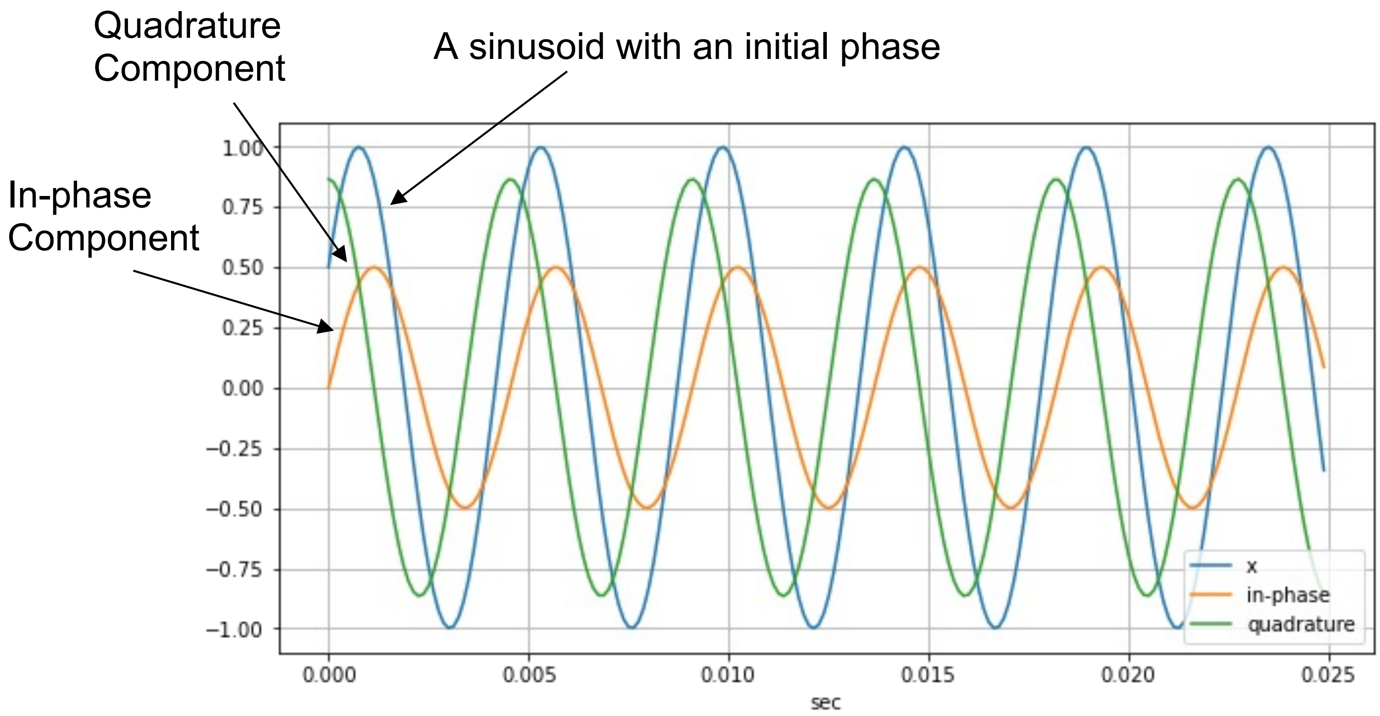
$$A_1 = A\sin(\phi)$$
$$A_2 = A\cos(\phi)$$

→

$$A = \sqrt{{A_1}^2 + {A_2}^2}$$
$$\phi = \tan^{-1}(A_1/A_2)$$

→ The peak amplitude and initial phase are determined by the amplitudes of the in-phase and quadrature components.

In-phase and Quadrature Sinusoidal Components



$$x(t) = \sin(2\pi \cdot f \cdot t + \phi)$$

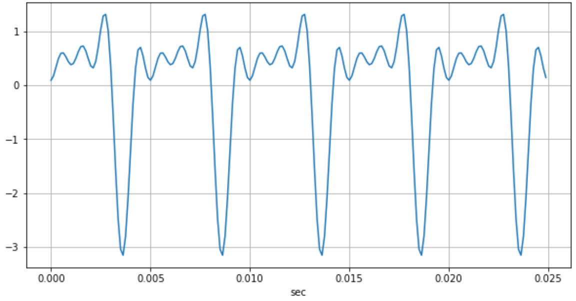
$$f = 220\ (Hz)$$

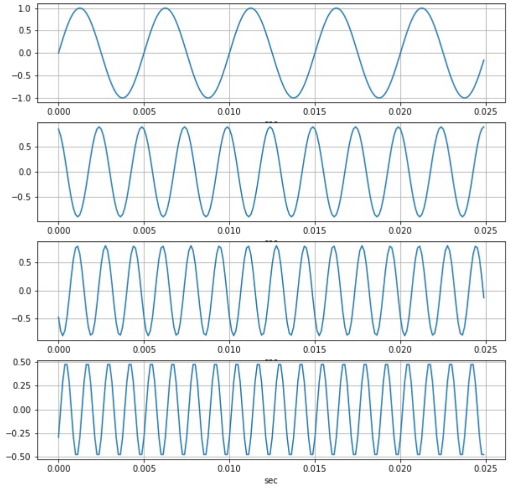
$$\phi = \pi/6$$

Fourier Series

- A periodic signal $x(t)$ with period P seconds can be represented with harmonic sinusoids

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{N} A_k \cos(\omega_k t - \phi_k) \qquad \omega_k = \frac{2\pi}{P}k \quad (k = 1, 2, \dots)$$





$$f_1 = \frac{1}{P}$$

$$f_2 = \frac{2}{P}$$

$$f_3 = \frac{3}{P}$$

$$f_4 = \frac{4}{P}$$

...

Harmonic frequencies

Fourier Series Coefficients

- A periodic signal $x(t)$ can be expressed with the in-phase and quadrature components

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{N} A_k \cos(\omega_k t - \phi_k) = \frac{A_0}{2} + \sum_{k=1}^{N} A_{1,k} \cos(\omega_k t) + A_{2,k} \sin(\omega_k t)$$

- The Fourier series coefficients $A_{1,k}$ and $A_{2,k}$ can be obtained by projecting $x(t)$ onto the in-phase and quadrature sinusoids

$$A_{1,k} = A_k \cos(\phi_k) = \frac{2}{P} \int_0^P x(t) \cos(\omega_k t)\, dt$$

$$A_{2,k} = A_k \sin(\phi_k) = \frac{2}{P} \int_0^P x(t) \sin(\omega_k t)\, dt$$

$$\longrightarrow \quad A_k = \sqrt{{A_{1,k}}^2 + {A_{2,k}}^2} \qquad \phi_k = \tan^{-1}(A_{2,k}/A_{1,k})$$

Complex Sinusoids

- Express the in-phase and quadrature sinusoid components as real and imaginary terms of complex numbers
  - The two components are regarded as a 2D vector

$$s(t) = s_{re}(t) + j \cdot s_{im}(t) = \cos(\omega t) + j \cdot \sin(\omega t)$$

- Euler's identity

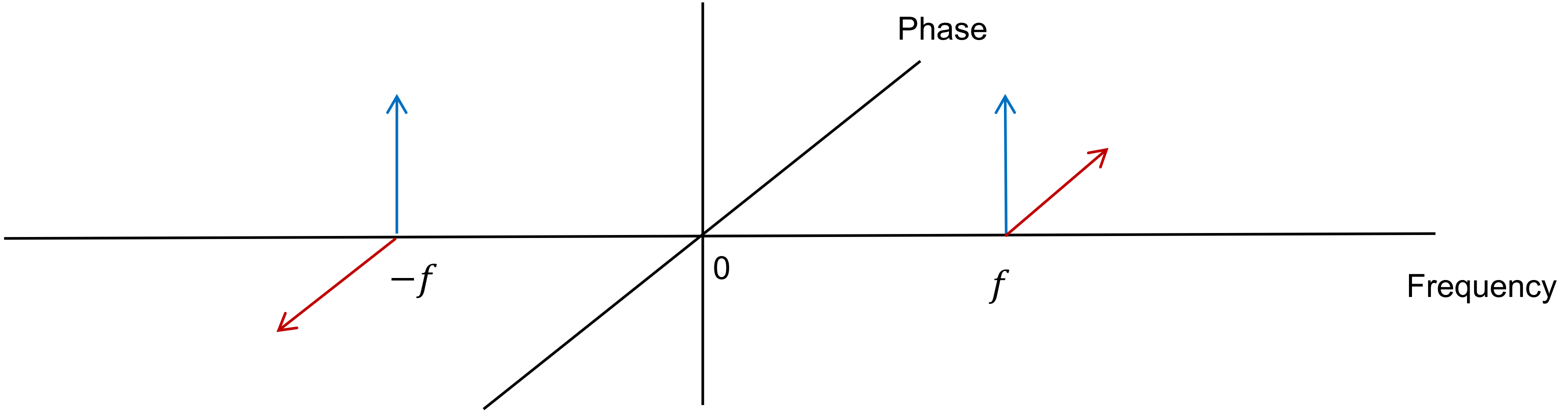
$$\boxed{\cos(\omega t) + j\sin(\omega t) = e^{j\omega t}}$$

  - This can be proved by Taylor's series
  - If $\omega t = \pi$, $e^{j\pi} + 1 = 0$ ("the most beautiful equation in math")

Complex Sinusoids

- Through the lens of complex sinusoid, cosine or sine functions always have **a pair of positive and negative frequencies**

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} = \frac{e^{j\omega t-\pi/2} - e^{-j\omega t+\pi/2}}{2}$$



Fourier Series with Complex Sinusoids

- A periodic signal $x(t)$ can be expressed with complex sinusoids

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{N} A_k \cos(\omega_k t - \phi_k) = \frac{A_0}{2} + \sum_{k=1}^{N} A_k \left(\frac{e^{j(\omega_k t - \phi_k)} + e^{-j(\omega_k t - \phi_k)}}{2}\right)$$

$$= \frac{A_0}{2} + \sum_{k=1}^{N} A_k \left(\frac{e^{j\omega_k t} e^{-j\phi_k} + e^{-j\omega_k t} e^{j\phi_k}}{2}\right) = \sum_{k=-N}^{N} C_k \, e^{j\omega_k t}$$

$$A_k e^{-j\phi_k}$$
$$= A_k \cos(\phi_k) - A_k \sin(\phi_k)$$
$$= A_{1,k} - jA_{2,k}$$

$$C_k = \begin{cases} \dfrac{A_0}{2} & (k = 0) \\ \dfrac{A_{1,k} - jA_{2,k}}{2} & (k > 0) \\ \dfrac{A_{1,-k} + jA_{2,-k}}{2} & (k < 0) \end{cases} \longrightarrow \boxed{C_k = \frac{1}{P}\int_0^P x(t) e^{-j\omega_k t}\, dt}$$

Fourier Series with Complex Sinusoids

$$(k > 0) \qquad \frac{A_{1,k} - jA_{2,k}}{2} = \frac{1}{2}\left(\frac{2}{P}\int_0^P x(t)\cos(\omega_k t)\,dt - j\frac{2}{P}\int_0^P x(t)\sin(\omega_k t)\,dt\right)$$

$$= \frac{1}{P}\left(\int_0^P x(t)(\cos(\omega_k t) - j\sin(\omega_k t))\,dt\right) = \frac{1}{P}\int_0^P x(t)e^{-j\omega_k t}\,dt$$

$$(k < 0) \qquad \frac{A_{1,-k} + jA_{2,-k}}{2} = \frac{1}{2}\left(\frac{2}{P}\int_0^P x(t)\cos(\omega_{-k} t)\,dt + j\frac{2}{P}\int_0^P x(t)\sin(\omega_{-k} t)\,dt\right)$$

$$= \frac{1}{P}\left(\int_0^P x(t)(\cos(\omega_k t) - j\sin(\omega_k t))\,dt\right) = \frac{1}{P}\int_0^P x(t)e^{-j\omega_k t}\,dt$$

$$(k = 0) \qquad \frac{A_0}{2} = \frac{1}{P}\int_0^P x(t)\,dt$$

To wrap up

- The Fourier Series Coefficients can be obtained as complex numbers

$$C_k = \frac{1}{P}\int_0^P x(t)e^{-j\omega_k t}\,dt$$

  - The peak amplitude of the k th sinusoid: $|C_k| = \sqrt{\mathrm{Re}\{C_k\}^2 + \mathrm{Im}\{C_k\}^2}$
  - The initial phase of the k th sinusoid: $\angle C_k = \tan^{-1}(\mathrm{Im}\{C_k\}/\mathrm{Re}\{C_k\})$

- The original signal can be reconstructed by the complex sinusoid with the complex coefficients

$$x(t) = \sum_{k=-N}^{N} C_k\, e^{j\omega_k t}$$

← $x(t)$ has both the positive and negative frequency components

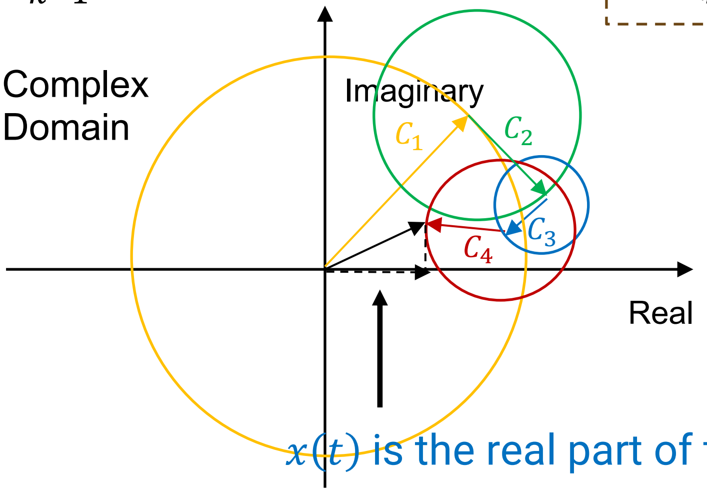
Understanding the Additive Sum

- A periodic signal $x(t)$ can be expressed with complex sinusoids

$$x(t) = \sum_{k=-N}^{N} C_k\, e^{j\omega_k t} = \sum_{k=-N}^{N} |C_k| e^{j\angle C_k}\, e^{j\omega_k t} = \sum_{k=-N}^{N} |C_k| e^{j(\omega_k t + \angle C_k)}$$

$$= C_0 + \sum_{k=1}^{N} \left(C_k e^{j\omega_k t} + \overline{C_k e^{j\omega_k t}}\right) = C_0 + 2\mathrm{Re}\left\{\sum_{k=1}^{N} C_k e^{j\omega_k t}\right\} = C_0 + 2\mathrm{Re}\left\{\sum_{k=1}^{N} |C_k| e^{j(\omega_k t + \angle C_k)}\right\}$$

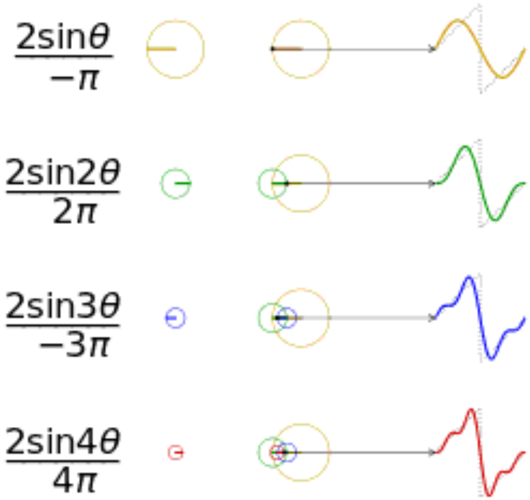
Projection onto the real axis



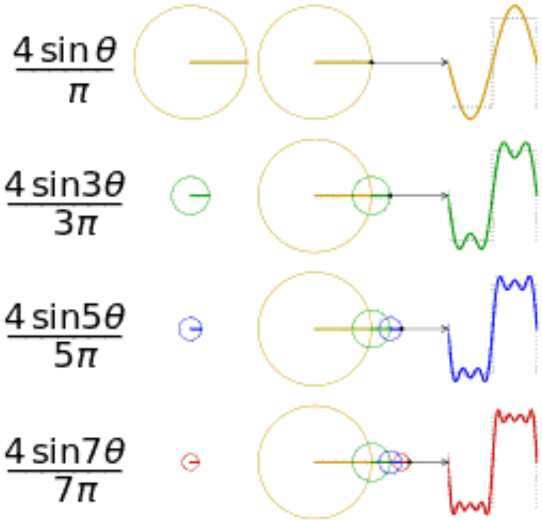
$x(t)$ is the real part of the sum of the rotating complex sinusoids

Additive Synthesis in the Complex Domain

- Sawtooth wave
  - $C_k = \frac{1}{k}(-1)^k \; (k = 1,2,3, \ldots)$



- Square wave
  - $C_k = \frac{1}{k} \; (k = 1,3,5, \ldots)$



Source: https://en.wikipedia.org/wiki/Fourier_series

Additive Synthesis in the Complex Domain

- Interactive Fourier Series and Additive synthesis
  - https://codepen.io/anon/pen/jPGJMK
  - https://teropa.info/harmonics-explorer/